Optimism and overconfidence investors' biases : a methodological note

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Introduction :

In financial literature, numerous biases of investor's behavior like overconfidence, optimism, conservatism, belief perseverance, anchoring or availability biases are underlined to explain financial decisions (Baker and *al.*, 2004). Nevertheless, to our minds, there is little evidence about their interactions and their respective roles on characteristic financial variables of cash-flows like the mean and variance variables (Barberis and Thaler, 2003) since these interactions appear very complex and contradictory in their effects on asset value.

Thus, the main objective of this paper is to elaborate operational tools in order to assess the impact of investors' biases on the mean and variance variables of assets like securities. As a consequence, we suppose that the mean and variance variables are representative of the asset value owned by an agent on the market and therefore are also representative of the agent or market's behavior[1].

In order to reach this objective, relationships between optimism and overconfidence biases have to be specified since in the financial literature the terms are used interchangeably or are linked with other biases. For instance, Heaton (2002) and Gervais and *al.* (2003) and Fairchild (2005) believe that these two biases are very linked to illusion of control of risky events and thus are very close. As a consequence, these biases are not very easy to define. Optimism (or pessimism) is usually defined in literature as a subjective overestimation (or underestimation) of favorable events due to psychological traits of decision-maker (Heaton, 2002 ; Gervais and *al.*, 2005) : as a consequence, optimism (pessimism) bias increases (decreases) expected returns ($\Delta\mu > 0$) but the impact on the variance is not very clear. Overconfidence (or underconfidence) bias represented mainly by α (with -1 < α < 1) relates to underestimation of the variance or risk of future events ($\Delta\sigma < 0$) by the agent due to the agent's overestimation (or underestimation) of his ability to affect the successful outcome of

[1] In the following of this paper, we use either the expressions "investor's behavior" or "market's behavior" since the investor (or the agent or again the decision-maker) is supposed to be representative of the market. In other terms, we suppose that every agent is the victim of biases with the same degree.

his project thanks to his private information (De Long and al., 1991 ; Daniel, Hirshleifer and Subrahmanyam, 1998 ; Odean, 1998 ; Gervais and Odean, 2001) ; but nothing is said about the impact on the expected return. In our work, we choose to study both biases together and to analyze their impacts on mean and variance at the same time by distinguishing two primary effects of biases : the first on probability of states (over or underconfidence) and the second on the level of cash-flows for each state (optimism or pessimism). In order to adopt a simple framework, we choose a binomial model (with only one period and two possible discrete states).

As a consequence, we define overconfidence from a directional point of view between two dates by the probability that either favorable state or unfavorable one occurs and optimism from a volatilistic point of view by the amount of the expected result for each state depending on the value of the variance. Thanks to this kind of model, we can take into account not only favorable or unfavorable scenario but also favorable or unfavorable asymmetric impact of biases on cash-flows and also different degrees of optimism and overconfidence biases.

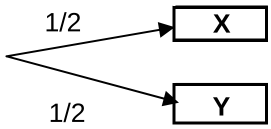
The main conclusion of this paper is the following one : we clarify the respective impact of the overconfidence and optimism biases on the mean and variance variables. This result will be very useful in the experimental financial framework in order to foresee for instance the impact of psychological traits on the assessment of a security and by aggregating the forecasts of the investors in order to evaluate a portfolio of assets. In order to achieve this result, the parameters of biases defined by the degree of overconfidence and optimism in favorable or unfavorable state are expressed independently the one from each other which allows us to describe their impact on cash-flows. Hence, the complex relation between biases, either combined or opposite, and their consequences on the investors' behavior are highlighted in this paper.

In the first part, the framework of the model is detailed. In a second part, the overconfidence bias is taken into account. In a third one, the optimism bias is introduced. In a next part, the overconfidence bias is combined with the optimism bias in order to have a more realistic model about investor's behavior. In an ultimate part, the decision-maker of the economic theory without any influence on the market equilibrium is given up in order to consider the case where only a part of investors are the victims of behavioral biases. It allows us to analyze other agent's behaviors like those of managers.

I Framework of the model

For investors, numerous financial situations are synonymous with binary choices between favorable and unfavorable states with probabilities for each. At the beginning, we assume there is no bias and we define the mean and the variance variables according to these two possible states. At equilibrium, the relation is supposed to be symetric and thus probabilities are equal. Different situations characterize this relation : cash-flows owing to projects, financial analyst forecasts, purchase or sale prices, …

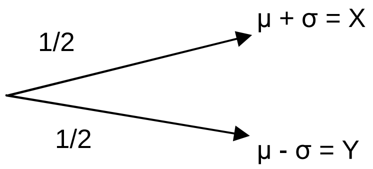
The process is the following one :



with the mean and the variance respectively : $\mu = \left(\frac{X+Y}{2}\right)$ and $\sigma^2 = \left(\frac{X-Y}{2}\right)^2$ (1)

[appendix 1]

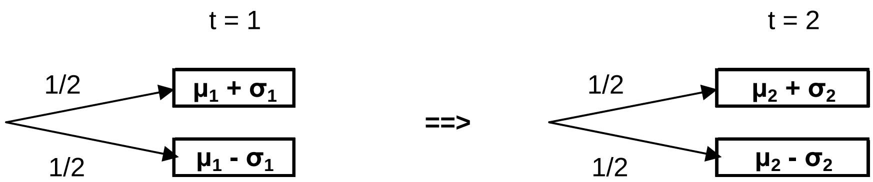
So, these expressions give the following values of X and Y :



(2)

[appendix 2]

The starting point of this paper consists in explaining the change between two dates of the mean and the variance of financial cash-flows as a result of two biases : the optimism and overconfidence biases of investors. On this basis, one can wonder about the transition between two dates at equilibrium (t = 1 and 2) :



by interpreting observed differences between means ($\Delta\mu = \mu_2 - \mu_1$) and variances ($\Delta\sigma = \sigma_2 - \sigma_1$).

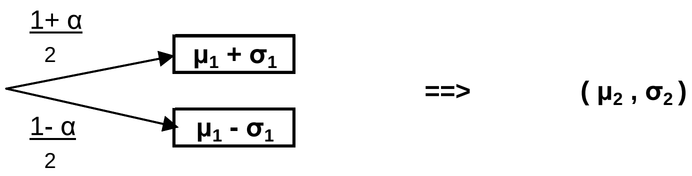
We shall try to explain the change from t=1 to t=2 by identifying two effects linked to investor's behavior :

- An effect of optimism (actions on possible states) linked to individual manager's characteristics
- An effect of overconfidence (actions on the probability of the occurrence of favorable state) linked to future states of nature,

and we shall assume that a consensus is found at the end of period.

II - Introduction of the overconfidence bias[2]

The *ex post* overconfidence bias (represented by α with -1 < α < 1) modifies the probability associated with favorable state. If α > 0, it represents situations of overconfidence and on the contrary case it represents situations of underconfidence of the market. Then, like in the model of Gervais and *al.* (2005), we assume[3] :



where relations turn out : $\mu_2 = \mu_1 + \alpha\sigma_1$ et $\sigma_2^2 = \left(1-\alpha^2\right)\sigma_1^2$ (3)

[appendix 3]

If it seems obvious that overconfidence increases the mean of μ_2, it is remarkable that this bias decreases the perceived variance by the investor. In fact, the overconfident decision-maker who experimented successful events tends to underestimate the risk *ex post*.

From the analysis of situations at equilibrium in t=1 and t= 2, one can deduce the level of confidence which characterizes the market :

$$\alpha = \frac{\mu_2 - \mu_1}{\sigma_1} = \begin{cases} \dfrac{\sqrt{\sigma_1^2 - \sigma_2^2}}{\sigma_1} & \text{if } \alpha > 0 \text{ : overconfidence} \\ -\dfrac{\sqrt{\sigma_1^2 - \sigma_2^2}}{\sigma_1} & \text{if } \alpha < 0 \text{ : lack of confidence} \end{cases} \quad (4)$$

with $-1<\alpha<+1$ and $\sigma_2 < \sigma_1$ according to appendix 3

[appendix 4]

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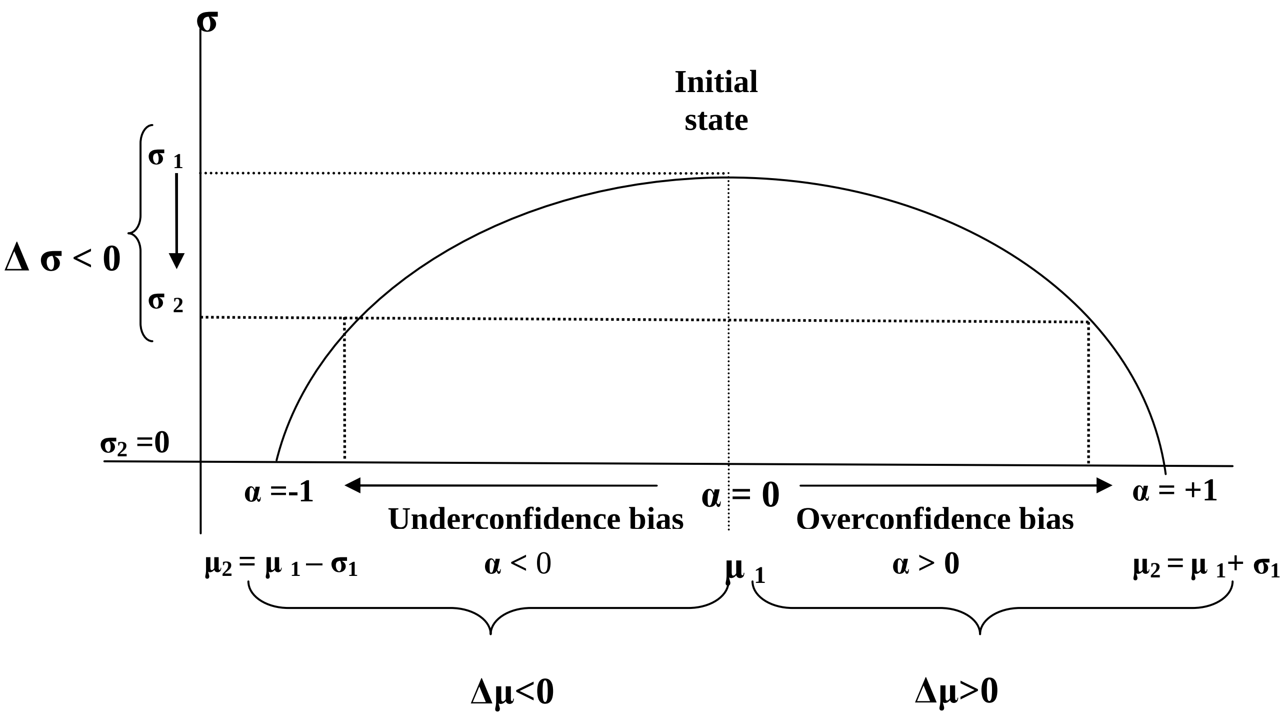
[2] The term of "overconfidence bias" refers indistinctly to α > 0 or α < 0 ; when it won't be the case, the sign of α will be precised.

[3] For instance, if α = 0,3 which characterizes the overconfidence bias, the favorable state occurs with a probability of $\left(\frac{1+0,3}{2} = \right)$ 65% whereas the unfavorable one with a probability of 35%.

One can notice from the previous equations that the overconfidence bias is positively related with the perceived difference between the two means $(\mu_2 - \mu_1)$ and also with the difference perceived between the two variances $(\sigma_1 - \sigma_2)$: the higher the overconfidence bias is, the more the decision-maker foresees a positive result and a weaker risk. We illustrate the impact of the overconfidence bias on mean and variance thanks to the following graph 1 :

Graph 1 : impact of the overconfidence bias on the mean and the variance



Two particular cases can be underlined :

- If $\Delta\sigma = 0$ or $\Delta\mu = 0$ then $\alpha = 0$: in other words, the decision-maker is no victim of biases since mean and variance are the same in the date 1 and 2 ;
- If $\Delta\sigma = -\sigma_1$ or $\sigma_2 = 0$ then $\alpha = 1$ or -1 : when variance equals zero, the decision-maker is certain about the result expected in date 2 ($\mu_1 + \sigma_1$ in favorable state or $\mu_1 - \sigma_1$ in unfavorable one).

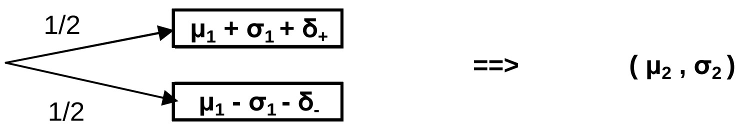
And two cases are possible :

- Case A "Return increases whereas risk decreases": since overconfidence bias increases, observed risk decreases (cf. appendix 3) and at the same time incites agent to invest and as a consequence increases the mean.
- Case B "Return and risk decrease": since underconfidence bias increases, risk decreases too since there is little chances that underconfident agent could size any profitable opportunity ; hence the mean decreases too.

Once we studied the overconfidence bias, the other bias of optimism is taken into account but in isolation at the beginning.

III Introduction of the optimism bias[4]

The optimism (or pessimism) bias modifies cash flows and reveals the belief in individual capacity illustrated for instance by the "hubris" of managers. The graph is then the following one :



Thus, the optimism bias is called δ_+ in favorable state and δ_- in the other state. The pessimism bias in unfavorable state characterizes the case of $\delta_- > 0$ since μ_2 decreases whereas in favorable state pessimism characterizes the case of $\delta_+ < 0$. The optimism bias in favorable state characterizes the case of $\delta_+ > 0$ and in unfavorable state the case of $\delta_- < 0$ since μ_2

[4] The term "optimism bias" is used whatever the state and the sign of δ_- or δ_+ . When it is not the case, the sign is précised.

increases. Thus, whatever the change of the risk, if $\Delta\mu>0$ then there is optimism whereas if $\Delta\mu<0$, there is pessimism.

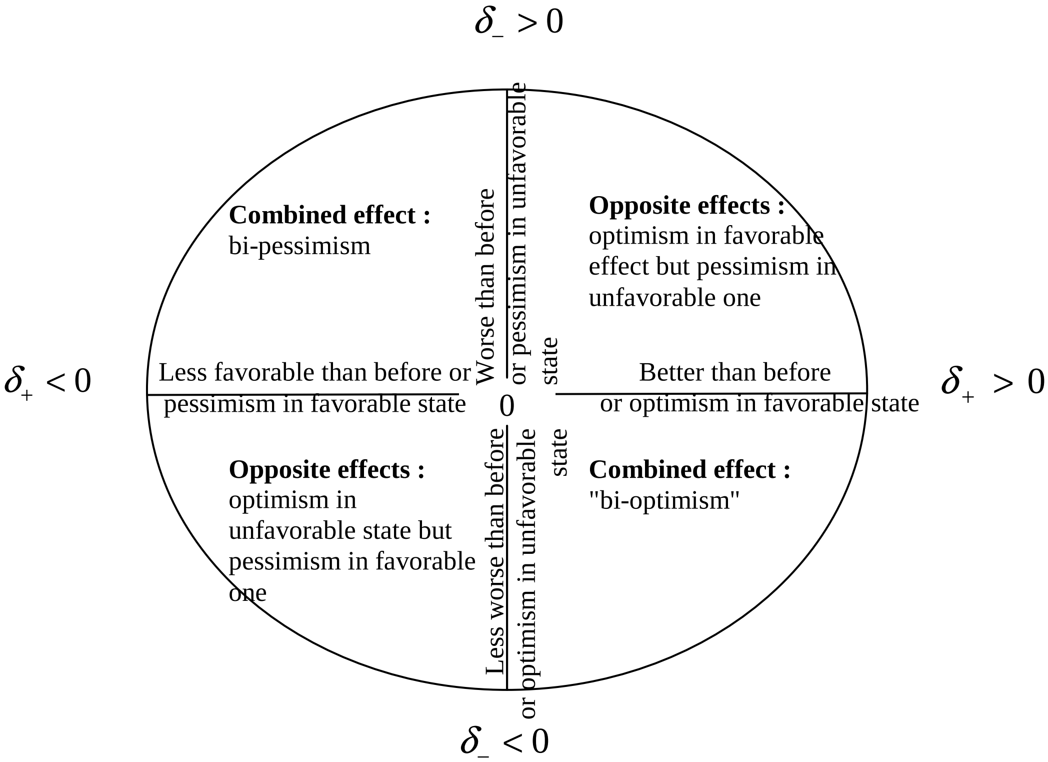
3 – 1 Situations of unilateral optimism

First, a decision-maker is supposed to be either solely optimistic (or pessimistic) on one of the two possible states and without bias on the other one. Different situations of unilateral optimism (or pessimism) can be highlighted which allows us to describe four possible of investor's behaviors:

- Optimism only in favorable situation ($\delta_+ > 0$ and $\delta_- = 0$)
- Optimism only in unfavorable situation ($\delta_+ = 0$ and $\delta_- < 0$)
- Pessimism only in favorable state ($\delta_+ < 0$ and $\delta_- = 0$)
- Pessimism only in unfavorable state ($\delta_+ = 0$ and $\delta_- > 0$)

These unilateral situations are extreme ones since in the reality intermediary situations are found that we called "bilateral biases". We can represent these four possible schemes on the graph 2 and qualify these situations:

Graph 2 : situations of unilateral or bilateral optimism bias



The four opposite or combined situations[5] which are intermediary situations are those that we study in the next paragraph.

3 – 2 Situations of bilateral biases

In fact, in order to introduce more realism in the analysis of decision-maker, all the possible schemes of the four previous situations can be detailed by admitting optimism (or pessimism) on one state and optimism (or pessimism) on the other one (with in index the effect on unfavorable state and in exponent the effect on favorable state) :

-Optimistic both in favorable and unfavorable situations ($\delta_+ > 0$ and $\delta_- < 0$) : B_O^O

[5] We use the term "combined" to illustrate the cases where there is optimism (or pessimism) biases both in favorable state and unfavorable one ($\delta_- < 0$ and $\delta_+ > 0$ or $\delta_- > 0$ and $\delta_+ < 0$). On the contrary, we use the term "opposite" when there is optimism in one state and pessimism in the other one ($\delta_- > 0$ and $\delta_+ < 0$ or $\delta_- < 0$ and $\delta_+ > 0$).

-Pessimistic both in favorable situation ($\delta_+ < 0$) and unfavorable one ($\delta_- > 0$) : B_P^P

-Optimistic in favorable situation ($\delta_+ > 0$) and pessimistic in unfavorable state ($\delta_- > 0$): B_P^O

-Pessimistic in favorable situation ($\delta_+ < 0$) and optimistic in bad situations ($\delta_- < 0$): B_o^P

Unilateral and bilateral situations can be summarized in the following table :

		Favorable state δ_+		
		< 0	*= 0*	*> 0*
Unfavorable state δ_-	*< 0*	B_o^P	U_O	B_O^O
	= 0	U^P	R	U^O
	> 0	B_P^P	U_P	B_P^O

R : perfect rational investor, B : bilateral bias and U : unilateral bias with in index the effect on unfavorable state and in exponent the effect on favorable state.

In order to preserve the consistency of the process, we assume the following constraint $(\sigma_1 + \delta_+) > -(\sigma_1 + \delta_-)$ or also $\frac{\delta_+ + \delta_-}{2} < -\sigma_1$ in order to make sure that favorable state is always better than unfavorable state.

The mean and the variance turn out :

$$\mu_2 = \mu_1 + \frac{\delta_+ - \delta_-}{2} \quad \text{and} \quad \sigma_2^2 = \left(\sigma_1 + \frac{\delta_+ + \delta_-}{2}\right)^2 \qquad (5)$$

[appendix 5]

Compared with the equation (1), one can notice that the impact of the optimism bias on the variance is more complex than the impact of the overconfidence bias since this latter bias always decreases the risk and increases (decreases) the mean if $\alpha > 0$ ($\alpha < 0$).

In fact, the optimism bias in favorable state ($\delta_+ > 0$) or the pessimism bias in the unfavorable state ($\delta_- > 0$) increase the risk since the probability is high that gains in favorable state and at the same time losses in unfavorable state are important. However, the impact on the mean is undecided.

The pessimism bias in favorable state ($\delta_+ < 0$) and the optimism bias in unfavorable one ($\delta_- < 0$) decrease the risk since gains and losses are more limited in both cases but the impact on the mean is always undecided.

However, the impact of the optimism bias on risk is undecided in all other cases ($\delta_+ < 0$ and $\delta_- > 0$ or $\delta_+ > 0$ and $\delta_- < 0$)[6] whereas this impact on the mean is direct since it is combined. In order to detail the complex impact of the optimism bias, we find the expression of optimism bias thanks to the equation (3) :

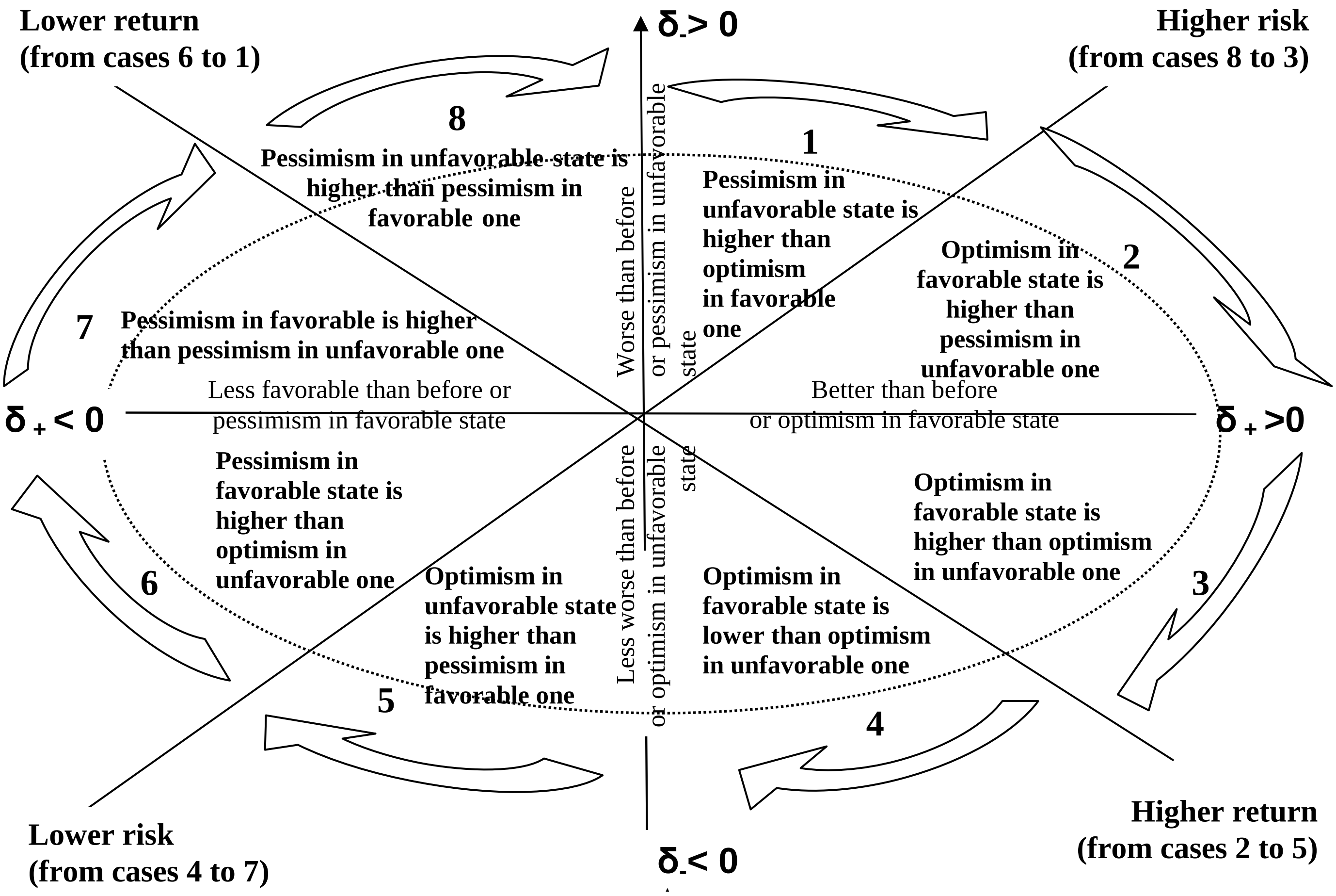
$$\delta_+ = (\sigma_2 - \sigma_1) + (\mu_2 - \mu_1) \quad \text{and} \quad \delta_- = (\sigma_2 - \sigma_1) - (\mu_2 - \mu_1) \qquad (6)$$

[appendix 6]

Then we can represent all the possible levels of optimism (or pessimism) either in favorable and unfavorable state and then describe 8 different situations according to three parameters μ, σ and δ by supposing $\alpha = 0$; the first bisecting line ($\Delta\mu = 0$) represents the frontier of optimism effects : on this line optimism and pessimism biases offsets themselves, below this line 4 cases describe all the possible situations of optimism. The second bisecting line ($\Delta\sigma = 0$) is the frontier of the sharing of the risk : above this line, the risk increases whatever optimism or pessimism bias :

[6] In the particular case where there is a same bias in favorable or unfavorable state ($\delta_+ = \delta_- = \delta$), one can notice that the mean is the same ($\mu_1 = \mu_2$) as if there were no bias. Moreover, variance increased (or decreased) by the amount of the optimism bias : $\sigma_2^2 = (\sigma_1 + \delta)^2$

<u>Graph 3 : impact of the optimism bias on the return and the risk</u>



One can remark that the intersection point of the four axes represents a rational decision-maker since the optimism bias equals to zero.

From case 2 to case 5, all possible situations of optimism are exposed since $(\mu_2 - \mu_1) > 0$. As far as it concerns the evolution of risk between the two dates, in case 2 and 3 risk increases because gains can vary a lot. On the contrary, in cases 4 and 5 risk decreases since neither losses nor gains can vary a lot.

From cases 6 to 1, all possible situations of pessimism are detailed with $(\mu_2 - \mu_1) < 0$. In cases 6 and 7, the risk decreases since losses cannot vary a lot whereas in cases 8 and 1, losses can vary a lot either in favorable or unfavorable state.

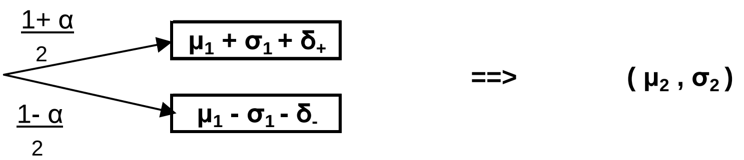
On the graph 3, we describe the situations we found on the graph 2 : in fact, cases 3 and 4 illustrate opposite effects on risk with higher total risk in case 3 but lower total risk in case 4 ; however, there are combined effects on the mean with higher total return. The cases 7 and 8 represent opposite effects on the risk with lower total risk in case 7 and higher total one in case 8 ; there are combined effects on the mean with lower total return. The cases 1 and 2 illustrate combined effects on risk with higher total risk but opposite effects on the return (higher return for case 2 and lower for case 1). In cases 5 and 6, whatever the state, the risk is lower whereas there are opposite effects on the return according to the state (total higher return for case 5 and total lower return in the case 6).

Up to now, we studied overconfidence and optimism biases separately, now both biases can be gathered into a same model.

IV - Taking into account optimism and overconfidence biases together

4 – 1 Expressions of the overconfidence and the optimism biases

With the combination of both biases, we have:



From rapid calculations, we obtain :

$$\mu_2 = \mu_1 + \frac{\delta_+ - \delta_-}{2} + \alpha\left(\sigma_1 + \frac{\delta_+ + \delta_-}{2}\right) \tag{7a}$$

$$\sigma_2^2 = \left(1-\alpha^2\right)\left(\sigma_1 + \frac{\delta_+ + \delta_-}{2}\right)^2 \tag{7b}$$

[appendix 7]

The extraction of bias parameters is direct for α :

$$\alpha = \frac{\mu_2 - \mu_1 - \frac{\delta_+ - \delta_-}{2}}{\sigma_1 + \frac{\delta_+ + \delta_-}{2}} \tag{8}$$

[appendix 8]

One can notice that it is possible to calculate couples of δ_+ and δ_- in function of α under constraints on bias [$-1 < \alpha < +1$ and $(\delta_+ + \delta_-) / 2 < -\sigma_1$] :

$$\begin{aligned} \delta_+ &= \sigma_2\sqrt{\frac{1-\alpha}{1+\alpha}} + (\mu_2 - \mu_1 - \sigma_1) \\ \delta_- &= \sigma_2\sqrt{\frac{1+\alpha}{1-\alpha}} - (\mu_2 - \mu_1 + \sigma_1) \end{aligned} \tag{9}$$

[appendix 9]

and extreme situations can be found for each :

$$\lim_{\alpha\to 1}\delta_+ = (\mu_2 - \mu_1 - \sigma_1) = \delta_+^{\min} \text{ and } \lim_{\alpha\to -1}\delta_+ = +\infty = \delta_+^{\max}$$

When the investor is sure that favorable state will occur (α = 1), then the optimism bias equals the positive change of mean minus the risk in the first period. This is consistent with equation (6) when $\sigma_2 = 0$.

$$\text{and } \lim_{\alpha\to 1}\delta_- = +\infty = \delta_-^{\max} \text{ and } \lim_{\alpha\to -1}\delta_- = -(\mu_2 - \mu_1 + \sigma_1) = \delta_-^{\min}$$

When the agent is sure that unfavorable state will occur (α = - 1) then $\sigma_2 = 0$ and the optimsim bias equals the change of the mean plus the risk of the first period : if the amount found is positive since there is a sign minus before the expression, it means that there is an optimism bias ($\delta_- < 0$) in unfavorable state.

From the equation (8), one can deduce that overconfidence increases with increasing mean. As far as it concerns the optimism bias (equation (9)), the relations between the mean, the variance and this bias is much more complex. In favorable state, the optimism bias increases not only with the positive change of the mean but also with the change of the variance. In this case, the overconfidence bias has an opposite effect on the optimism bias : the higher the overconfidence bias is, the less important the influence of the variation of risk is. In unfavorable state, optimism bias depends on positive change of the mean since δ_- is more negative. The overconfidence bias has a combined effect on the optimism bias : the higher the overconfidence bias is, the more important the influence of the variation of risk is[7].

4 – 2 Interaction of the overconfidence and the optimism bias on mean and variance

Since we have found the mathematical expressions of each bias, we can wonder how both biases interact with the mean and variance variables. In order to isolate the effect of the overconfidence and optimism biases on the mean, the graph 3 can be used again in order to take into account α. By supposing $\Delta\mu = 0$ and from the equation (7a), the optimism bias in unfavorable state can be deduced :

[7] A particular case offers an explicit solution when $\delta_+ = \delta_- = \delta$:

$$\mu_2 = \mu_1 + \alpha(\sigma_1 + \delta) \quad \text{and} \quad \sigma_2^2 = (1-\alpha^2)(\sigma_1+\delta)^2$$

The parameters of biases are then the following ones:

$$\alpha = \frac{\mu_2 - \mu_1}{\sqrt{\sigma_2^2 + (\mu_2-\mu_1)^2}} \quad \text{and} \quad \delta = \sqrt{\sigma_2^2 + (\mu_2-\mu_1)^2} - \sigma_1$$

[appendix 10]

We can interpret these expressions by asserting logically that the more perceived difference of means ($\mu_2 - \mu_1$) is positive, the greater is α. In the same way, the more positive the perceived difference of variances $(\sigma_2 - \sigma_1)$ is, the more optimistic the decision-maker is. It is also interesting to note that both biases are independent from each other.

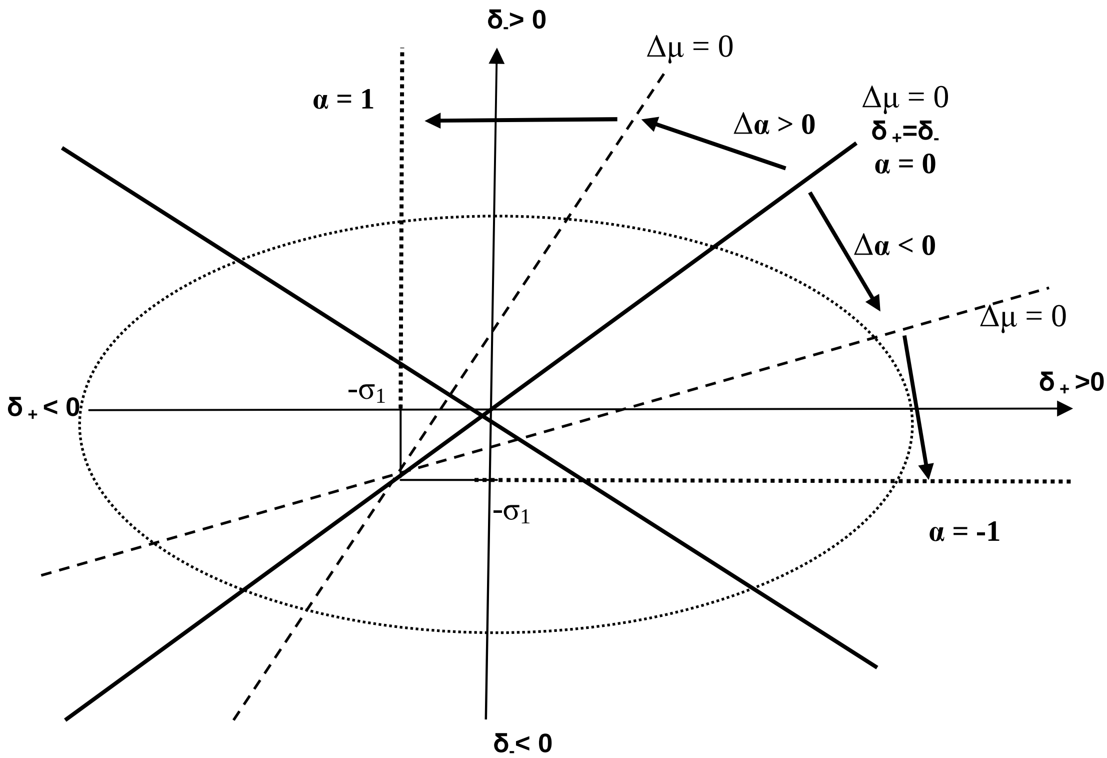
$$\delta_- = \frac{1+\alpha}{1-\alpha}\delta_+ + \frac{2\alpha\sigma_1}{1-\alpha} \qquad (10a)$$

When $\alpha = 1$, then optimism bias in unfavorable state is infinite and when $\alpha = -1$, $\delta_- = -\sigma_1$. These results are consistent with those obtained from extreme limits of equation (9) when $\Delta\mu = 0$.

$$\text{The slope is : } a = \frac{1+\alpha}{1-\alpha} \text{ with } \begin{cases} a > 1 \text{ when } \alpha > 0 \\ \text{and} \\ 0 < a < 1 \text{ when } \alpha < 0 \end{cases}$$

Equation (10a) is represented in the following graph 4 by a dotted line above or under the first bisecting line according to the α value.

<u>Graph 4 : Impact of the overconfidence and optimism biases on the mean</u>



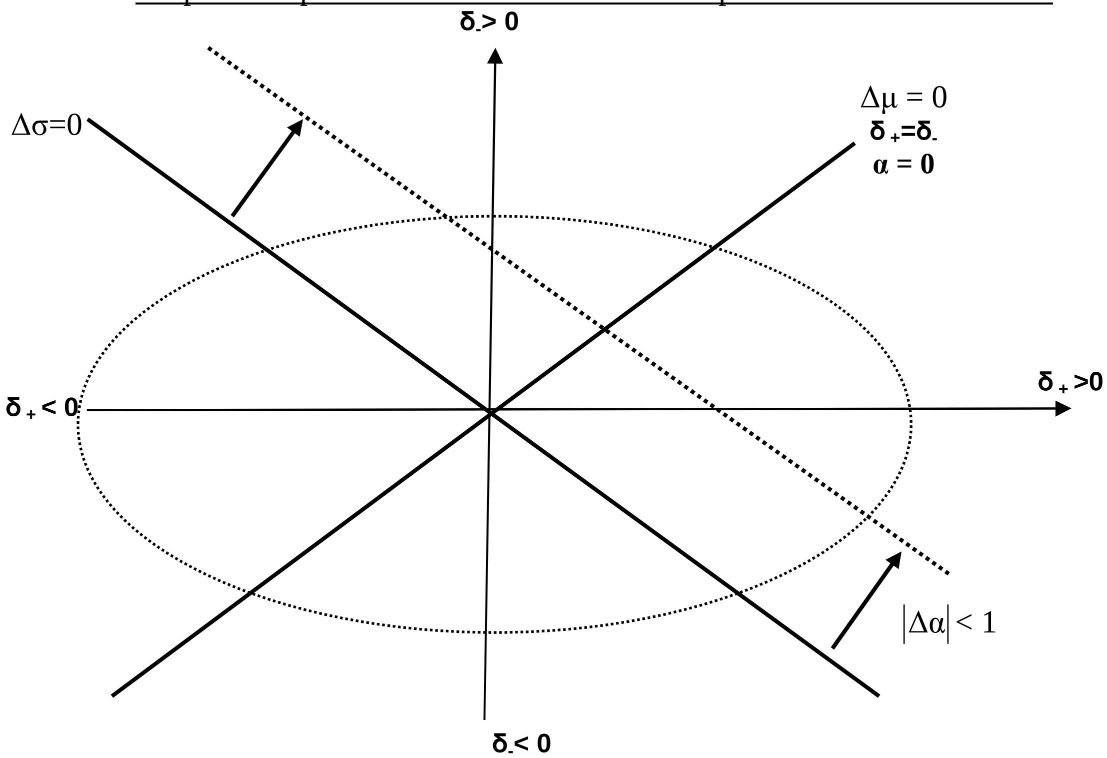
When the overconfidence bias increases (α > 0), the pessimism bias in unfavorable state ($\delta_- >$ 0) is unlikely to be higher than the optimism bias in favorable state ($\delta_+ > 0$). In other terms, cases where optimism in favorable state (Δμ >0) are higher than pessimism in unfavorable state will be more numerous. On the contrary, when the overconfidence bias decreases, situations with optimism bias in favorable state are supposed to be less numerous.

From equation (7b) and by supposing that Δσ=0, we deduce :

$$\delta_- = 2\sigma\left[\frac{1}{\sqrt{(1-\alpha^2)}}-1\right]-\delta_+ \tag{10b}$$

The slope is a constant equal to -1 and since the constant $2\sigma\left[\frac{1}{\sqrt{(1-\alpha^2)}}-1\right]$ is always positive, the equation (10b) represented by the dotted line on graph 5 can move only towards the top of the graph :

Graph 5 : Impact of the overconfidence and optimism biases on the risk

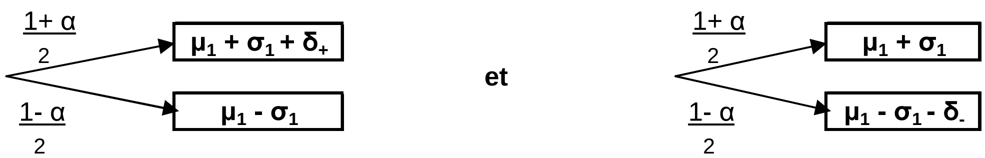


On this graph, one can observe that the area where risk increases is less important. This is consistent with the equation (7b) for which when the overconfidence bias varies the risk decreases by the same amount both in favorable and unfavorable state.

4 – 3 In unilateral frame

To solve residual ambiguity of previous framework, we suggest restraining the case where optimism situations are not combined (either δ_+ or δ_-).

Optimism bias "on the top" ($\delta_- = 0$) Optimism bias "on the bottom" ($\delta_+ = 0$)



with $\delta_+ \geq -2\sigma_1$ and $\delta_- \leq -2\sigma_1$

The mean and the variance can be deduced from previous equations (7) with $\delta_- = 0$ in the first case and $\delta_+ = 0$ in the other one :

case with δ_+ case with δ_-

$$\mu_2 = \mu_1 + \alpha\sigma_1 + \frac{\delta_+}{2}(1+\alpha) \quad \text{and} \quad \mu_2 = \mu_1 + \alpha\sigma_1 - \frac{\delta_-}{2}(1-\alpha)$$

$$\sigma_2^2 = (1-\alpha^2)\left(\sigma_1 + \frac{\delta_+}{2}\right)^2 \quad \text{and} \quad \sigma_2^2 = (1-\alpha^2)\left(\sigma_1 + \frac{\delta_-}{2}\right)^2$$

In favorable state, as we said before, overconfidence and optimism biases increase the mean due to a cumulative effect. However, if overconfidence bias decreases the risk, optimism bias increases it.

In unfavorable state, the overconfidence and optimism biases ($\delta_- < 0$) increase both the mean and decrease both the variance since losses cannot be important.

Thanks to the previous expressions of the mean and the variance, we can explicitly extract the parameters of overconfidence and optimism biases independently from each other :

case with δ_+ case with δ_-

$$\alpha = \frac{(\mu_2 - \mu_1 + \sigma_1)^2 - \sigma_2^2}{(\mu_2 - \mu_1 + \sigma_1)^2 + \sigma_2^2} \quad \text{and} \quad \alpha = \frac{\sigma_2^2 - (\mu_1 - \mu_2 + \sigma_1)^2}{\sigma_2^2 + (\mu_1 - \mu_2 + \sigma_1)^2}$$

$$\delta_+ = \frac{\sigma_2^2}{\mu_2 - \mu_1 + \sigma_1} + \mu_2 - \mu_1 - \sigma_1 \quad \text{and} \quad \delta_- = \frac{\sigma_2^2}{\mu_1 - \mu_2 - \sigma_1} - \mu_2 + \mu_1 - \sigma_1 \tag{11}$$

[appendix 11]

Again, the more positive perceived difference of mean ($\mu_2 - \mu_1$) is, the greater is α whatever the states. If $\sigma_2 = 0$, then $\alpha = 1$ or -1 according to the state : this result is consistent with the graph 1. In the same way, the more positive the perceived difference of variance is, the more optimistic the decision-maker is. If $\sigma_2 = 0$, the expressions of optimism biases are consistent with those found thanks to the equation (9) : if the situation at date 2 is sure, then optimism bias in favorable state equals μ_2 minus μ_1 minus σ_1.

4 – 4 Numerical examples

In order to illustrate the interest of our study, the particular situation for which μ and σ increase between two dates has been chosen : at the date one we suppose that we have (μ_1=10.00, σ_1=3.00) and in date 2 (μ_2=11.40 , σ_2=4.41). From previous equation 7, since μ and σ are given in date 1 and 2, we find for different values of α all the possible combinations of δ_- and δ_+ which explain the change of the mean and the variance from the date 1 to date 2 and thus illustrate the complex relationship between both biases of optimism and overconfidence :

(μ_1 , σ_1) 10.00 , 3.00		A	B	C	D	E	(μ_2 , σ_2) 11.40 , 4.41
	α	-0,6	0	0,200	0,303	0,767	
	δ_+	7,21	2,809	2,000	1,626	0	
	δ_-	-2,19	0,009	1,000	1,626	7,750	

Thus, the change from an equilibrium (μ_1, σ_1) towards another type of equilibrium (μ_2, σ_2) is possible from the three variable biases (α, δ_+, δ_-). On the basis of the couples (μ_1=10.00, σ_1=3.00) and biases characterized for instance by (α=0.200, δ_+=2.0 and δ_-=1.0), we obtain the couple (μ_2=11.40 , σ_2=4.41).

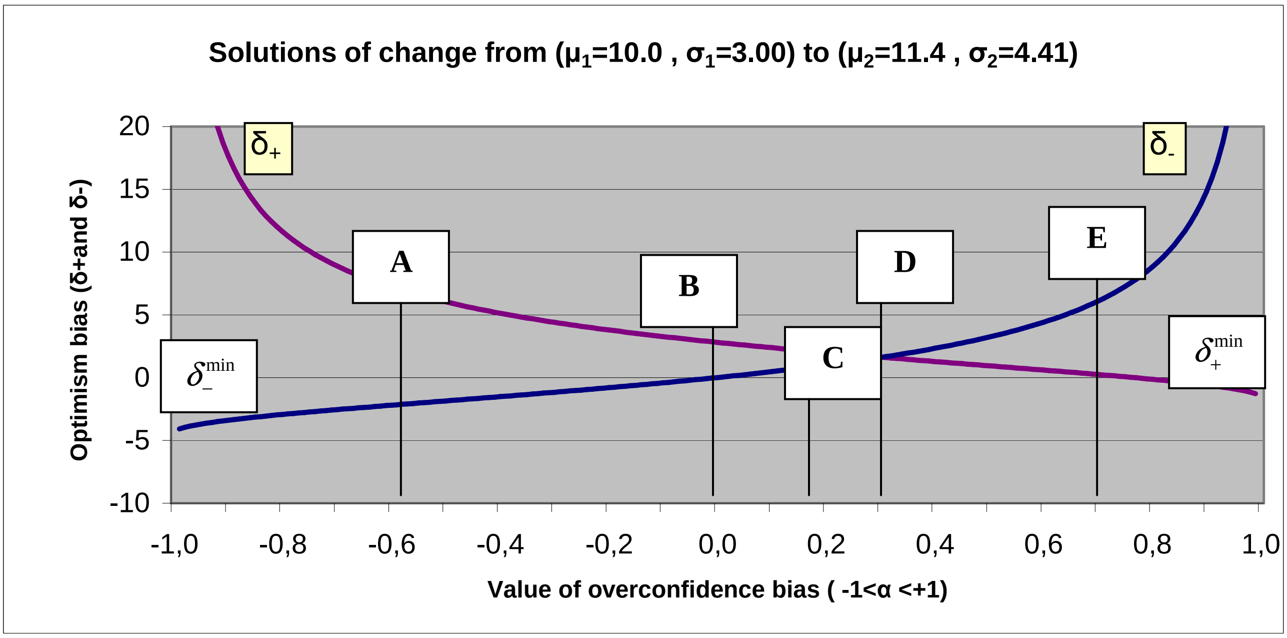
In the case C, investors are overconfident and at the same time optimism in favorable state since they evaluate with a probability of 60% [(1+ α)/2 =1,2/2] that favorable state will reach 15 (X= μ_1+ σ_1+ δ_+=10+3+2) instead of 10 in initial situation. But they are pessimistic in unfavorable one since they evaluate with a probability of 40% ([(1- α)/2 = 0,8/2] that unfavorable state will reach only 6 (Y= μ_1- σ_1- δ_-=10-3-1) instead of 10. We can explain the improvement of the situation by saying that there is a combined effect of the overconfidence bias and optimism bias in favorable state which is higher than pessimism in unfavorable one.

Numerous other cases can be described which lead to the same result in period 2 :

- **Case E** : investors are both very overconfident and very pessimistic in unfavorable state since they estimate with only a probability of 11,6% [(1- α)/2 = (1-,767)/2] that the unfavorable state will decrease up to -0,75 (Y= μ_1- σ_1- δ_-=10-3-7,75) and with a 88,4% probability that the favorable state will be at 13 (X= μ_1+ σ_1+ δ_+=10+3+0). We can say that the high degree of overconfidence in favorable state offsets the high degree of pessimism in unfavorable state
- **Case D** : investors are both overconfident (expected favorable state occurs with a probability of 65%) and equally allocate optimism and pessimism biases on both states of processes (±1,626) : the optimism bias increases the risk.

- **Case B** : investors are not the victims of the overconfidence bias but they are very optimistic in favorable state since $X = \mu_1 + \sigma_1 + \delta_+ = 10+3+2,809 = 15,809$ and a few pessimistic in unfavorable one since $Y = \mu_1 - \sigma_1 - \delta_- = 10-3-0,009 = 6,991$. The optimism (or pessimism) bias explains the increasing level of risk.
- **Case A** : investors are very underconfident since they estimate that the favorable state will occur with a weak probability of 20% but they are very optimistic whatever the states because $Y = \mu_1 - \sigma_1 - \delta_- = 10-3-(-2,19) = 9,19$ and $X = \mu_1 + \sigma_1 + \delta_+ = 10+3+7,21 = 20,21$. Optimism bias offsets underconfidence bias but leds to increasing risk.

The generalization to all the solutions is direct[8] :



Thanks to the extreme values of equation (9), the minimum values of the optimism bias are :

$$\delta_-^{min} = -(\mu_2 - \mu_1 + \sigma_1) = -(11,4 - 10 + 3) = -4,4 \text{ and } \delta_+^{min} = -(11,4 - 10 + 3) = -1,6$$

[8] The case where the mean and the variance decrease has been realized and is in appendix 12. Only minimum values of δ_- and δ_+ (cf. equation 8) changed.

And, even in the particular case where parameters μ and σ remain constant from date 1 to date 2, we notice that there are always solutions characterized by the three variables [α, δ_+, δ_-]. Thus, if there is no change for μ and σ (μ = 10.0 and σ=3.0), we can detail all the possible combinations of bias parameters in order to describe all behaviors leading to the same consequences (including the obvious solution without bias i.e. the absolute rationality of the decision-maker (α=0, δ_+=0, δ_-=0))

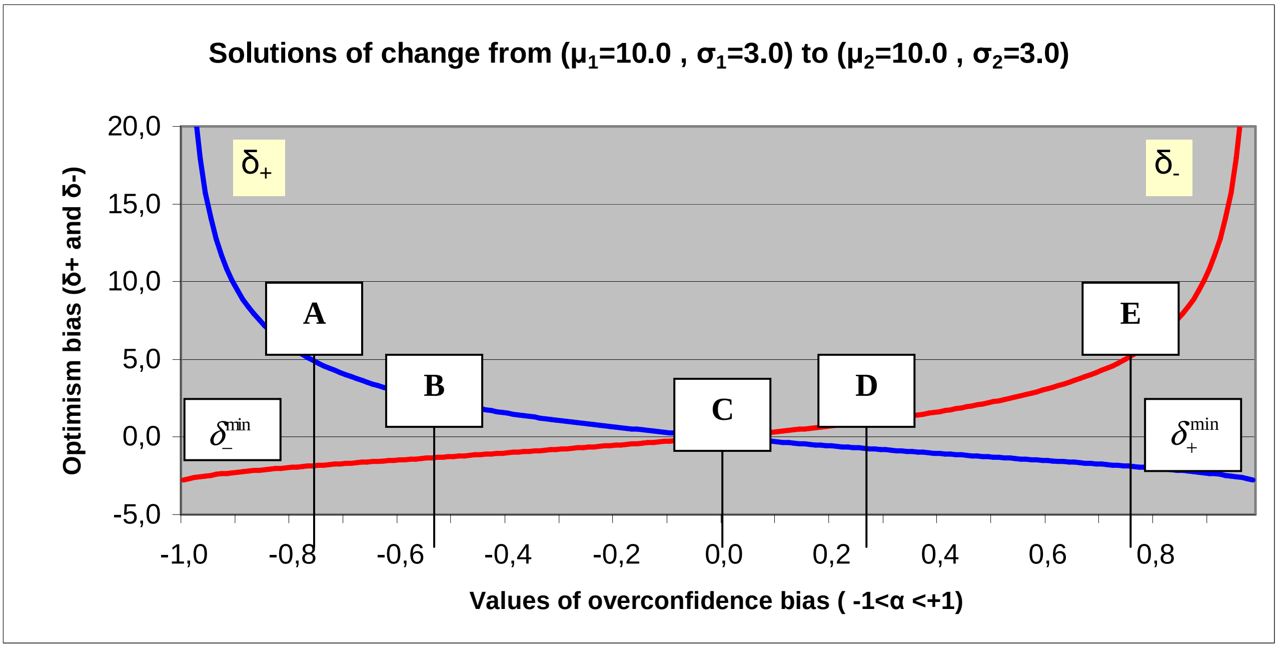
Some situations illustrate the previous remarks:

(μ_1 , σ_1)							**(μ_2 , σ_2)**
10.0 , 3.0		**A**	**B**	**C**	**D**	**E**	10.0 , 3.0
	α	-0,8	-0,6	0,0	0,28	0,8	
	δ_+	2,0	1,5	0,0	-0,75	-2,0	
	δ_-	-6,0	-3,000	0,0	-1,000	6,0	

- **Cases A and B** : investors are "highly optimistic" with a negative overconfidence bias. We have bi-optimism situation ($\delta_+ > 0$ and $\delta_- < 0$ that is optimism on both favorable and unfavorable states) as if optimism offsets negative overconfidence in order to remain mean and variance of project the same in date 1 and 2 ; moreover the risk remains the same since optimism in favorable state increases the risk whereas optimism in unfavorable state decreases it.
- **Case D** : investor is "risk averse" and overconfident. It is an asymmetrical situation since the optimism bias in unfavorable state increases the risk but less than the pessimistic bias in favorable one which decreases the risk.
- **Case E** : investor is very confident since he believes the favorable state will occur with 90% [(1+ α)/2 =0,9]. At the same time, the situation is symmetrical because he is "highly pessimistic" in both states.

One can notice that it is not possible to find uni-optimism situations (that is optimism in either favorable or unfavorable state)

Again, the generalization to all the solutions is direct:



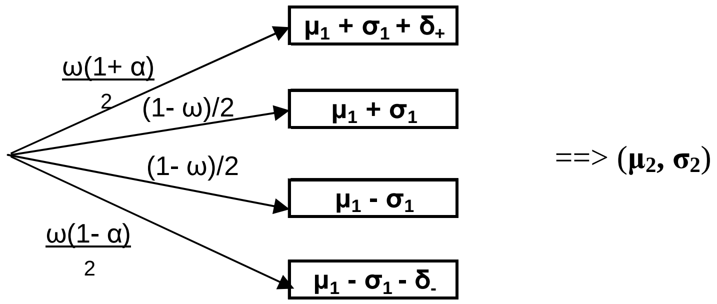
Again, the extreme values are the following ones :

$$\delta_-^{min} = -(\mu_2 - \mu_1 + \sigma_1) = -(10 - 10 + 3) = -3 \text{ and } \delta_+^{min} = -(10 - 10 + 3) = -3$$

Thus, for a complex bilateral optimistic situation with overconfidence bias, we can use a mathematical tool to give an operational dimension to the problem. Research in behavioral finance should give some helps.

V Taking into account only a proportion of investors (ω) being the victims of biases

On the basis of the equilibrium situation in t=1 (μ_1 , σ_1), only a proportion ω of investors is supposed to be the victims of biases. The situations can be illustrated according to the amount of cash-flows earned with or without both biases :



Since favorable state has always to be better than the unfavorable one, the following condition is necessary: $(\sigma_1 + \delta_+) > -(\sigma_1 + \delta_-)$

From the previous graph, one can deduce the mathematical expressions of mean and variance parameters in period 2 :

$$\mu_2 = \mu_1 + \omega\left(\frac{\delta_+ - \delta_-}{2}\right) + \omega\alpha\left[2\sigma_1 + \left(\frac{\delta_+ + \delta_-}{2}\right)\right] \qquad (13)$$

and :

$$\sigma_2^2 = \left(\frac{\omega(1+\alpha)}{2}\right)[\mu_1 + \sigma_1 + \delta_+]^2 + \left(\frac{1-\omega}{2}\right)[\mu_1 + \sigma_1]^2 + \left(\frac{1-\omega}{2}\right)[\mu_1 - \sigma_1]^2 + \left(\frac{\omega(1-\alpha)}{2}\right)[\mu_1 - \sigma_1 - \delta_-]^2 - \mu_2^2$$

[appendix 13]

The resolution of this system depends on 4 parameters α, ω, δ_+ et δ_- since couples (μ_1,σ_1) and (μ_2,σ_2) are the same as before.

The mathematical resolution of the system requires to set couples (α, ω) under the following conditions :

- $-1 < \alpha < +1$
- $0 < \omega < +1$
- $(\sigma_1 + \delta_+) > -(\sigma_1 + \delta_-)$

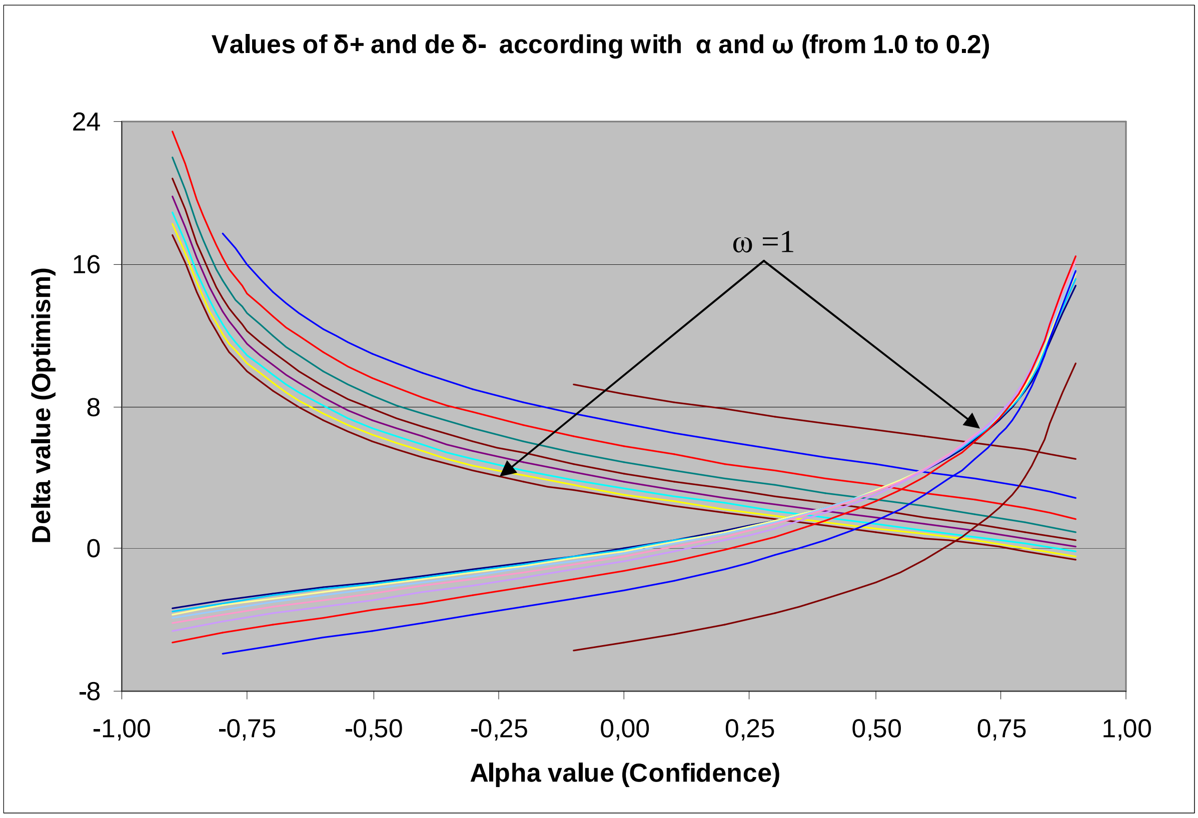
and δ_- can be deduced from μ_2 (equation 13) :

$$\delta_- = \left[\frac{\mu_2 - \mu_1 - \omega\left(\delta_+ \left(\frac{1+\alpha}{2} \right) + \alpha\sigma_1 \right)}{\omega\left(\frac{\alpha - 1}{2} \right)} \right]$$

When the same values of (μ_1,σ_1) and (μ_2,σ_2) are taken up, the simulations show that some combinations cannot be possible especially when ω tends towards 0.

One can represent all the combinations in a graph with ω the proportion of investors in favorable state (decreasing graph) and or in unfavorable state (increasing graph)[9] :

[9] The data which allowed us to realize the following graph can be read in appendix 14.



One remarks that in the particular case where ω = 1, all the combinations of δ_+ and δ_- found in the previous part are represented.

The main conclusions are the following ones :

- the impact of the proportion of investors being the victims of biases on the degree of optimism (or pessimism) is relatively weak except perhaps for the smallest proportion (ω =0.2)
- as long as investors are underconfident (α< 0), bi-optimism situations are observed for which investors are called "highly optimistic" ;
- when investors are overconfident, they are always optimistic in favorable state but in unfavorable state they are either optimistic ("highly optimistic") or pessimistic according to the degree of overconfidence. This is true all the more so since ω is important.

In the particular case where $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$ and from the expression of μ_2, we observe that optimism biases do not depend from ω but only from :

$$\alpha = \left[\frac{2\sigma_1 + \delta_+ + \delta_-}{\delta_+ - \delta_-} \right] \qquad (15)$$

[appendix 15]

Thus, studies which aim at calculating behavior biases of investors can only determine one of the three parameters (overconfidence α, optimism at the top δ_+ and at the bottom δ_-) and the proportion of individuals concerned (ω). Otherwise, for lack of limited measures, limiting the scope of solutions such, for instance, a situation of overconfidence ($\alpha>0$) is possible.

Conclusion:

A methodological approach has been proposed in this paper in order to have a comprehensive mathematical expression of the impact of behavior biases on mean and variance values. In other words, this study deals with how both biases are mentally integrated by decision-makers in their assessment of securities.

The relationship between optimism and overconfidence biases has been clarified and has allowed us to detail how these biases influence the mean and the variance variables. The combination of two biases provides much richer methodological implications than examining overconfidence in isolation. If it seems obvious that the overconfidence bias decreases the risk and optimism (or pessimism) increases (decreases) the mean, the impact of overconfidence on the mean and the influence of the optimism bias on the risk is complex. Our paper tried to clarify these impacts.

In the particular case where we taking into account only a part of investors being the victims of biases, the respective role of optimism and overconfidence biases according to the favorable or unfavorable state has not be modified. Finally, our research could be expanded by taking account measures of overconfidence proposed by Malmendier and Tate (2005) in the particular case of a proportion of managers is the victim of behavioral biases.

Empirically and in the framework of experimental finance, our work could be useful in order to aggregate the investor's believes which would allow us to give the value of an asset. In other words, explaining how the optimism and overconfidence biases influence the market or agent's behaviour would be the objective. In fact, the steps of the experience would be the following ones : first, we shall give each individual the mean and variance values of a security ; second, we answer them to give their forecasts about the probability that favorable (and unfavorable) state occurs and about the estimation of optimism bias both in favorable and unfavorable states ; third, from the values of both biases, we could assess the new values of mean and variance and thus the value of the security. But the link between the values of biases and the values of mean and variance is far from being obvious. That is why our paper has given an expression of this link on a mathematical basis.

After studying how behavioral biases influence an asset value, the final step would be the aggregation of these values to assess a portfolio of securities.

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APPENDIX

- **Appendix 1 :**

$$\mu = \frac{1}{2}X + \frac{1}{2}Y = \left(\frac{X+Y}{2}\right)$$

$$\sigma^2 = \frac{1}{2}\left(X^2\right) + \frac{1}{2}\left(Y^2\right) - \mu^2 = \left(\frac{X^2+Y^2}{2}\right) - \left(\frac{X+Y}{2}\right)^2$$

$$= \left(\frac{2X^2 + 2Y^2 - X^2 - Y^2 - 2XY}{4}\right) = \left(\frac{X-Y}{2}\right)^2$$

- **Appendix 2 :**

$$\mu + \sigma = \left(\frac{X+Y}{2}\right) + \left(\frac{X-Y}{2}\right) = \left(\frac{X+Y+X-Y}{2}\right) = X$$

$$\mu - \sigma = \left(\frac{X+Y}{2}\right) - \left(\frac{X-Y}{2}\right) = \left(\frac{X+Y-X+Y}{2}\right) = Y$$

- **Appendix 3 :**

$$\mu_2 = \left(\frac{1+\alpha}{2}\right)(\mu_1 + \sigma_1) + \left(\frac{1-\alpha}{2}\right)(\mu_1 - \sigma_1) = \frac{1}{2}\left\{(\mu_1 + \sigma_1 + \mu_1 - \sigma_1) + \alpha(\mu_1 + \sigma_1 - \mu_1 + \sigma_1)\right\}$$

$$\mu_2 = \frac{2\mu_1 + \alpha 2\sigma_1}{2} = \mu_1 + \alpha\sigma_1$$

$$\sigma_2^2 = \left(\frac{1+\alpha}{2}\right)(\mu_1 + \sigma_1)^2 + \left(\frac{1-\alpha}{2}\right)(\mu_1 - \sigma_1)^2 - \mu_2^2$$

$$= \frac{1}{2}\left\{\left(\mu_1^2 + \sigma_1^2 + 2\mu_1\sigma_1 + \mu_1^2 + \sigma_1^2 - 2\mu_1\sigma_1\right) + \alpha\left(\mu_1^2 + \sigma_1^2 + 2\mu_1\sigma_1 - \mu_1^2 - \sigma_1^2 + 2\mu_1\sigma_1\right)\right\} - \mu_2^2$$

$$= \frac{1}{2}\left\{\left(2\mu_1^2 + 2\sigma_1^2\right) + \alpha\left(4\mu_1\sigma_1\right)\right\} - \mu_1^2 - \alpha^2\sigma_1^2 - 2\alpha\mu_1\sigma_1 = \mu_1^2 + \sigma_1^2 - \mu_1^2 - \alpha^2\sigma_1^2 + 2\alpha\mu_1\sigma_1 + 2\alpha\mu_1\sigma_1$$

$$\sigma_2^2 = \sigma_1^2\left(1 - \alpha^2\right)$$

- **Appendix 4 :**

$$\mu_2 = \mu_1 + \alpha\sigma_1 \Rightarrow \alpha = \frac{\mu_2 - \mu_1}{\sigma_1}$$

By construction, the probability to reach a state is between 0 and 1 thus -1 < α < +1

$$\sigma_2^2 = \sigma_1^2\left(1 - \alpha^2\right) \Rightarrow \alpha^2 = 1 - \frac{\sigma_2^2}{\sigma_1^2} = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2}$$

$$\alpha = \frac{\mu_2 - \mu_1}{\sigma_1} = \begin{cases} \dfrac{\sqrt{\sigma_1^2 - \sigma_2^2}}{\sigma_1} & \text{if } 0 < \alpha < 1 \\ -\dfrac{\sqrt{\sigma_1^2 - \sigma_2^2}}{\sigma_1} & \text{if } -1 < \alpha < 0 \end{cases}$$

- **Appendix 5 :**

$$\mu_2 = \frac{1}{2}\{\mu_1 + \sigma_1 + \delta_+ + \mu_1 - \sigma_1 - \delta_-\} = \mu_1 + \frac{\delta_+ - \delta_-}{2}$$

$$\sigma_2^2 = \frac{1}{2}\left\{(\mu_1 + \sigma_1 + \delta_+)^2 + (\mu_1 - \sigma_1 - \delta_-)^2\right\} - \mu_2^2$$

$$= \frac{1}{2}\left\{\left(\mu_1^2 + \sigma_1^2 + \delta_+^2 + 2\mu_1\sigma_1 + 2\mu_1\delta_+ + 2\sigma_1\delta_+\right) + \left(\mu_1^2 + \sigma_1^2 + \delta_-^2 - 2\mu_1\sigma_1 - 2\mu_1\delta_- + 2\sigma_1\delta_-\right)\right\} - \mu_2^2$$

$$= \frac{1}{2}\left(2\mu_1^2 + 2\sigma_1^2 + \left(\delta_+^2 + \delta_-^2\right) + 2\mu_1\left(\delta_+ - \delta_-\right) + 2\sigma_1\left(\delta_+ + \delta_-\right)\right) - \mu_1^2 - \left(\frac{\delta_+ - \delta_-}{2}\right)^2 - 2\mu_1\left(\frac{\delta_+ - \delta_-}{2}\right)$$

$$= \sigma_1^2 + \frac{\delta_+^2 + \delta_-^2}{2} + \sigma_1\left(\delta_+ + \delta_-\right) - \left(\frac{\delta_+^2 + \delta_-^2 - 2\delta_+\delta_-}{4}\right) = \sigma_1^2 + \sigma_1\left(\delta_+ + \delta_-\right) + \left(\frac{\delta_+ + \delta_-}{2}\right)^2$$

$$\sigma_2^2 = \left(\sigma_1 + \frac{\delta_+ + \delta_-}{2}\right)^2$$

- **Appendix 6 :**

$$\mu_2 = \mu_1 + \left(\frac{\delta_+ - \delta_-}{2}\right) \Rightarrow \delta_+ = 2\left(\mu_2 - \mu_1\right) + \delta_-$$

$$\sigma_2^2 = \left(\sigma_1 + \frac{\delta_+ - \delta_-}{2}\right)^2 \Rightarrow \left(\frac{\delta_+ + \delta_-}{2}\right) = \sigma_2 - \sigma_1 \Rightarrow \delta_- = 2\left(\sigma_2 - \sigma_1\right) - \delta_+$$

$$\delta_+ = 2\left(\mu_2 - \mu_1\right) + 2\left(\sigma_2 - \sigma_1\right) - \delta_+$$
$$\delta_+ = \left(\mu_2 - \mu_1\right) + \left(\sigma_2 - \sigma_1\right)$$

$$\delta_- = 2\left(\sigma_2 - \sigma_1\right) - \left(\mu_2 - \mu_1\right) + \left(\sigma_2 - \sigma_1\right)$$
$$\delta_- = \left(\sigma_2 - \sigma_1\right) - \left(\mu_2 - \mu_1\right)$$

- **Appendix 7 :**

$$\mu_2 = \left(\frac{1+\alpha}{2}\right)(\mu_1+\sigma_1+\delta_+) + \left(\frac{1-\alpha}{2}\right)(\mu_1-\sigma_1-\delta_-)$$

$$= \frac{1}{2}\left\{(\mu_1+\sigma_1+\delta_+ + \mu_1-\sigma_1-\delta_-) + \alpha(\mu_1+\sigma_1+\delta_+ - \mu_1+\sigma_1+\delta_-)\right\}$$

$$\mu_2 = \mu_1 + \frac{\delta_+-\delta_-}{2} + \alpha\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)$$

$$\sigma_2^2 = \left(\frac{1+\alpha}{2}\right)\left(\mu_1+\sigma_1+\delta_+-\mu_1-\frac{\delta_+-\delta_-}{2}-\alpha\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)\right)^2$$

$$+\left(\frac{1-\alpha}{2}\right)\left(\mu_1-\sigma_1-\delta_--\mu_1-\frac{\delta_+-\delta_-}{2}-\alpha\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)\right)^2$$

$$= \frac{1}{2}\left\{(1+\alpha)\left[\sigma_1(1-\alpha)-\frac{2\delta_-}{2}-\frac{\delta_+}{2}+\frac{\delta_-}{2}-\alpha\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)\right]^2 + (1-\alpha)\left[-\sigma_1(1+\alpha)-(1+\alpha)\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)\right]^2\right\}$$

$$= \frac{1}{2}\left\{(1+\alpha)\left[(1-\alpha)\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)\right]^2 + (1-\alpha)\left[-(1+\alpha)\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)\right]^2\right\}$$

$$= \frac{1}{2}\left\{(1-\alpha^2)\left[(1-\alpha)\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)^2 + (1+\alpha)\left[-\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)^2\right]\right]\right\}$$

$$= (1-\alpha^2)\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)^2\left[\frac{1-\alpha+1+\alpha}{2}\right]$$

$$\sigma_2^2 = (1-\alpha^2)\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)^2$$

- **Appendix 8 :**

$$\mu_2 = \mu_1 + \frac{\delta_+-\delta_-}{2} + \alpha\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right) \Rightarrow \alpha = \frac{\mu_2-\mu_1-\frac{\delta_+-\delta_-}{2}}{\sigma_1+\frac{\delta_++\delta_-}{2}}$$

$$\sigma_2^2 = (1-\alpha^2)\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)^2 = \left(\frac{\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)^2 - \left(\mu_2-\mu_1-\frac{\delta_+-\delta_-}{2}\right)^2}{\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)^2}\right)\left(\sigma_1+\frac{\delta_++\delta_-}{2}\right)^2$$

$$\sigma_2^2=\left(\sigma_1+\frac{\delta_+ +\delta_-}{2}-\mu_2+\mu_1+\frac{\delta_+ -\delta_-}{2}\right)\left(\sigma_1+\frac{\delta_+ +\delta_-}{2}+\mu_2-\mu_1-\frac{\delta_+ -\delta_-}{2}\right)$$

$$\sigma_2^2=\left(\sigma_1-\mu_2+\mu_1+\delta_+\right)\left(\sigma_1+\mu_2-\mu_1+\delta_-\right)$$

- **Appendix 9 :**

$$\sigma_2^2=\left(1-\alpha^2\right)\left(\sigma_1+\frac{\delta_+ +\delta_-}{2}\right)^2 \Rightarrow \left(\frac{\delta_+ +\delta_-}{2}\right)=\frac{\sigma_2}{\sqrt{(1-\alpha^2)}}-\sigma_1$$

$$\mu_2=\mu_1+\frac{\delta_+ -\delta_-}{2}+\alpha\left(\sigma_1+\frac{\delta_+ +\delta_-}{2}\right)\Rightarrow$$

$$\left(\frac{\delta_+ -\delta_-}{2}\right)=\mu_2-\mu_1-\alpha\sigma_1-\alpha\left[\frac{\sigma_2}{\sqrt{(1-\alpha^2)}}-\sigma_1\right]=(\mu_2-\mu_1)-\alpha\left[\frac{\sigma_2}{\sqrt{(1-\alpha^2)}}\right]$$

$$\delta_+=\left(\frac{\delta_+ +\delta_-}{2}\right)+\left(\frac{\delta_+ -\delta_-}{2}\right)=\frac{\sigma_2\,(1-\alpha)}{\sqrt{(1-\alpha^2)}}+(\mu_2-\mu_1-\sigma_1)=\sigma_2\sqrt{\frac{1-\alpha}{1+\alpha}}+(\mu_2-\mu_1-\sigma_1)$$

$$\delta_-=\left(\frac{\delta_+ +\delta_-}{2}\right)-\left(\frac{\delta_+ -\delta_-}{2}\right)=\frac{\sigma_2\,(1+\alpha)}{\sqrt{(1-\alpha^2)}}-(\mu_2-\mu_1-\sigma_1)=\sigma_2\sqrt{\frac{1+\alpha}{1-\alpha}}-(\mu_2-\mu_1+\sigma_1)$$

- **Appendix 10 :**

$$\mu_2=\mu_1+\alpha\left(\sigma_1+\delta\right)\Rightarrow \alpha=\frac{\mu_2-\mu_1}{\sigma_1+\delta}$$

$$\sigma_2^2=\left(1-\alpha^2\right)\left(\sigma_1+\delta\right)^2=\left(1-\left(\frac{\mu_2-\mu_1}{\sigma_1+\delta}\right)^2\right)\left(\sigma_1+\delta\right)^2=\left(\frac{(\sigma_1+\delta)^2-(\mu_2-\mu_1)^2}{(\sigma_1+\delta)^2}\right)\left(\sigma_1+\delta\right)^2$$

$$\sigma_2^2=\left(\sigma_1+\delta\right)^2-\left(\mu_2-\mu_1\right)^2\Rightarrow\left(\sigma_1+\delta\right)=\sqrt{\sigma_2^2+\left(\mu_2-\mu_1\right)^2}$$

$$\delta=\sqrt{\sigma_2^2+\left(\mu_2-\mu_1\right)^2}-\sigma_1$$

$$\alpha=\frac{\mu_2-\mu_1}{\sigma_1+\sqrt{\sigma_2^2+\left(\mu_2-\mu_1\right)^2}-\sigma_1}=\frac{\mu_2-\mu_1}{\sqrt{\sigma_2^2+\left(\mu_2-\mu_1\right)^2}}$$

- **Appendix 11 :**

Optimism bias "on the top"

$$\left.\begin{array}{l}\mu_2=\mu_1+\alpha\sigma_1+\dfrac{\delta_+}{2}(1+\alpha)\Rightarrow\alpha=\dfrac{\mu_2-\mu_1-\dfrac{\delta_+}{2}}{\sigma_1+\dfrac{\delta_+}{2}}\\ \sigma_2^2=(1-\alpha^2)\left(\sigma_1+\dfrac{\delta_+}{2}\right)^2\end{array}\right\}\Rightarrow\sigma_2^2=\left[\frac{\left(\sigma_1+\dfrac{\delta_+}{2}\right)^2-\left(\mu_2-\mu_1-\dfrac{\delta_+}{2}\right)^2}{\left(\sigma_1+\dfrac{\delta_+}{2}\right)^2}\right]\left(\sigma_1+\frac{\delta_+}{2}\right)^2$$

$$\sigma_2^2=\left[\left(\sigma_1+\frac{\delta_+}{2}\right)-\left(\mu_2-\mu_1-\frac{\delta_+}{2}\right)\right]\cdot\left[\left(\sigma_1+\frac{\delta_+}{2}\right)+\left(\mu_2-\mu_1-\frac{\delta_+}{2}\right)\right]=[\sigma_1+\delta_+-\mu_2+\mu_1]\cdot[\sigma_1+\mu_2-\mu_1]$$

$$\Rightarrow\delta_+=\left(\frac{\sigma_2^2}{\mu_2-\mu_1+\sigma_1}+\mu_2-\mu_1-\sigma_1\right)$$

$$\alpha=\frac{\mu_2-\mu_1-\dfrac{\delta_+}{2}}{\sigma_1+\dfrac{\delta_+}{2}}=\frac{\mu_2-\mu_1-\dfrac{1}{2}\left(\dfrac{\sigma_2^2}{\mu_2-\mu_1+\sigma_1}+\mu_2-\mu_1-\sigma_1\right)}{\sigma_1+\dfrac{1}{2}\left(\dfrac{\sigma_2^2}{\mu_2-\mu_1+\sigma_1}+\mu_2-\mu_1-\sigma_1\right)}=\frac{\dfrac{\mu_2}{2}-\dfrac{\mu_1}{2}+\dfrac{\sigma_1}{2}-\dfrac{1}{2}\left(\dfrac{\sigma_2^2}{\mu_2-\mu_1+\sigma_1}\right)}{\dfrac{\mu_2}{2}-\dfrac{\mu_1}{2}+\dfrac{\sigma_1}{2}+\dfrac{1}{2}\left(\dfrac{\sigma_2^2}{\mu_2-\mu_1+\sigma_1}\right)}$$

$$\alpha=\frac{(\mu_2-\mu_1+\sigma_1)^2-\sigma_2^2}{(\mu_2-\mu_1+\sigma_1)^2+\sigma_2^2}$$

Optimism bias "on the bottom"

$$\left.\begin{array}{l}\mu_2=\mu_1+\alpha\sigma_1-\dfrac{\delta_-}{2}(1+\alpha)\Rightarrow\alpha=\dfrac{\mu_2-\mu_1+\dfrac{\delta_-}{2}}{\sigma_1+\dfrac{\delta_-}{2}}\\ \sigma_2^2=(1-\alpha^2)\left(\sigma_1+\dfrac{\delta_-}{2}\right)^2\end{array}\right\}\Rightarrow\sigma_2^2=\left[\frac{\left(\sigma_1-\dfrac{\delta_-}{2}\right)^2-\left(\mu_2-\mu_1+\dfrac{\delta_-}{2}\right)^2}{\left(\sigma_1+\dfrac{\delta_-}{2}\right)^2}\right]\left(\sigma_1+\frac{\delta_-}{2}\right)^2$$

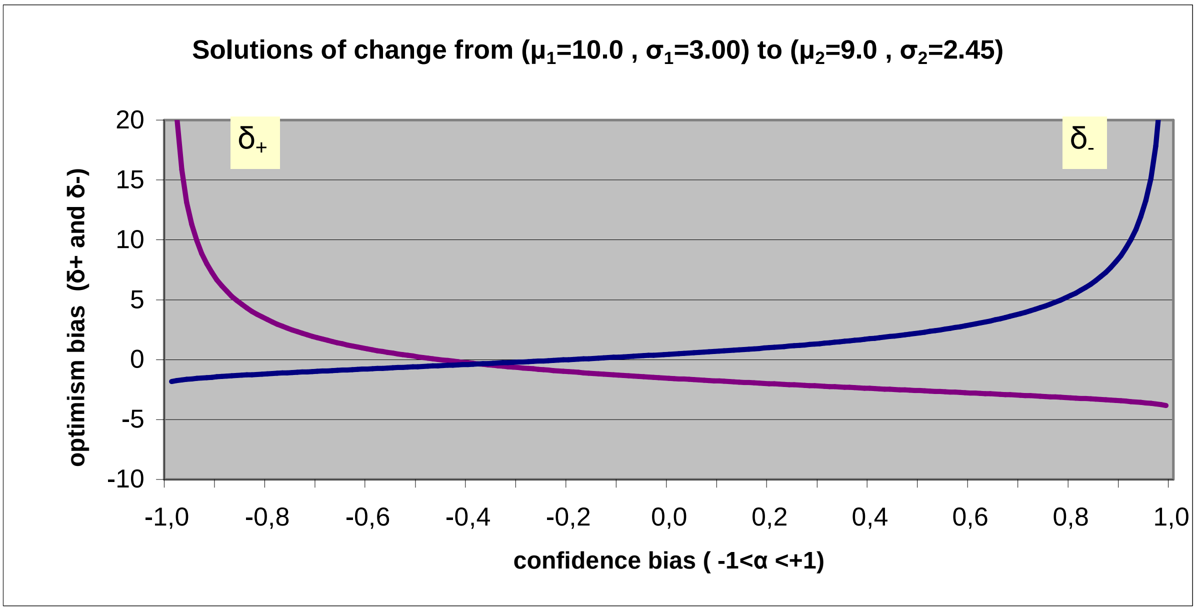
$$\sigma_2^2=\left[\left(\sigma_1-\frac{\delta_-}{2}\right)-\left(\mu_2-\mu_1+\frac{\delta_-}{2}\right)\right]\cdot\left[\left(\sigma_1-\frac{\delta_-}{2}\right)+\left(\mu_2-\mu_1+\frac{\delta_-}{2}\right)\right]=[\sigma_1-\delta_--\mu_2+\mu_1]\cdot[\sigma_1+\mu_2-\mu_1]$$

$$\Rightarrow\delta_-=\left(\frac{\sigma_2^2}{\mu_1-\mu_2-\sigma_1}-\mu_2+\mu_1-\sigma_1\right)$$

$$\alpha = \frac{\mu_2 - \mu_1 + \frac{\delta_-}{2}}{\sigma_1 + \frac{\delta_-}{2}} = \frac{\mu_2 - \mu_1 + \frac{1}{2}\left(\frac{\sigma_2^2}{\mu_1 - \mu_2 + \sigma_1} - \mu_2 + \mu_1 - \sigma_1\right)}{\sigma_1 + \frac{1}{2}\left(\frac{\sigma_2^2}{\mu_1 - \mu_2 + \sigma_1} - \mu_2 + \mu_1 - \sigma_1\right)} = \frac{\frac{\mu_2}{2} - \frac{\mu_1}{2} - \frac{\sigma_1}{2} + \frac{1}{2}\left(\frac{\sigma_2^2}{\mu_1 - \mu_2 + \sigma_1}\right)}{\frac{\mu_1}{2} - \frac{\mu_2}{2} + \frac{\sigma_1}{2} - \frac{1}{2}\left(\frac{\sigma_2^2}{\mu_1 - \mu_2 + \sigma_1}\right)}$$

$$\alpha = \frac{\sigma_2^2 - (\mu_1 - \mu_2 + \sigma_1)^2}{\sigma_2^2 + (\mu_1 - \mu_2 + \sigma_1)^2}$$

- **Appendix 12 : The case where mean and variance decrease**



(μ_1, σ_1) (10.0 , 3.0) → (μ_2, σ_2) (9.0 , 2.45)

	A	B	C	D	E
α	0,200	-0,455	-0,378	0	-0,200
δ_+	-2,000	0	-0,354	-1,551	-1
δ_-	1,000	-0,500	-0,354	0,449	0

- **Appendix 13 :**

$$\mu_2 = \left(\frac{\omega(1+\alpha)}{2}\right)[\mu_1 + \sigma_1 + \delta_+] + \left(\frac{1-\omega}{2}\right)[\mu_1 + \sigma_1] + \left(\frac{1-\omega}{2}\right)[\mu_1 - \sigma_1] + \left(\frac{\omega(1-\alpha)}{2}\right)[\mu_1 - \sigma_1 - \delta_-]$$

$$= \frac{1}{2}\left\{(\mu_1 + \sigma_1 + \mu_1 + \sigma_1) + \omega\left[(\mu_1 + \sigma_1 + \delta_+ - \mu_1 - \sigma_1 - \mu_1 - \sigma_1 + \mu_1 - \sigma_1 - \delta_-) + \alpha(\mu_1 + \sigma_1 + \delta_+ - \mu_1 + \sigma_1 + \delta_-)\right]\right\}$$

$$\mu_2 = \mu_1 + \omega\left(\frac{\delta_+ - \delta_-}{2}\right) + \omega\alpha\left[2\sigma_1 + \left(\frac{\delta_+ + \delta_-}{2}\right)\right]$$

$$\sigma_2^2 = \left(\frac{\omega(1+\alpha)}{2}\right)[\mu_1 + \sigma_1 + \delta_+]^2 + \left(\frac{1-\omega}{2}\right)[\mu_1 + \sigma_1]^2 + \left(\frac{1-\omega}{2}\right)[\mu_1 - \sigma_1]^2 + \left(\frac{\omega(1-\alpha)}{2}\right)[\mu_1 - \sigma_1 - \delta_-]^2 - \mu_2^2$$

Appendix 14 :

		μ1	10,0	σ1	3,00		μ2	11,4	σ2	4,41	
		ω									
		1,0	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1
	α										
δ+	-0,90	17,619	18,223	18,932	19,775	20,786	22,010	23,462	na	na	na
	-0,50	6,037	6,371	6,770	7,255	7,858	8,625	9,625	10,923	na	na
	-0,10	3,274	3,544	3,869	4,269	4,775	5,433	6,324	7,575	9,186	na
	0,00	2,809	3,068	3,381	3,766	4,255	4,895	5,768	7,011	8,691	na
	0,10	2,388	2,638	2,939	3,312	3,785	4,409	5,265	6,501	8,243	na
	0,50	0,946	1,161	1,424	1,752	2,176	2,742	3,542	4,753	6,708	na
	0,90	-0,588	-0,408	-0,188	0,094	0,463	0,969	1,709	2,893	5,077	na
δ-	-0,90	-3,388	-3,520	-3,688	-3,907	-4,204	-4,631	-5,291	na	na	na
	-0,50	-1,854	-1,951	-2,077	-2,248	-2,492	-2,858	-3,458	-4,581	na	na
	-0,10	-0,412	-0,474	-0,562	-0,689	-0,881	-1,191	-1,735	-2,833	-5,757	na
	0,00	0,009	-0,043	-0,119	-0,234	-0,412	-0,705	-1,232	-2,322	-5,309	na
	0,10	0,474	0,434	0,370	0,270	0,108	-0,167	-0,676	-1,758	-4,814	na
	0,50	3,237	3,260	3,272	3,255	3,193	3,027	2,625	1,591	-1,876	na
	0,90	14,819	15,128	15,432	15,784	16,131	16,416	16,475	15,633	10,461	na

- **Appendix 15 :**

$$\mu_2 = \mu_1 + \omega\left(\frac{\delta_+ - \delta_-}{2}\right) + \omega\alpha\left[\sigma_1 + \left(\frac{\delta_+ + \delta_-}{2}\right)\right] \quad \text{and if } \mu_2 = \mu_1$$

$$\text{then} \quad 0 = \omega\left(\frac{\delta_+ - \delta_-}{2}\right) + \omega\alpha\left[\sigma_1 + \left(\frac{\delta_+ + \delta_-}{2}\right)\right] \quad \text{and}$$

$$\alpha = \left[\frac{2\sigma_1 + \delta_+ + \delta_-}{\delta_+ - \delta_-}\right]$$